### Introduction

Shear fluidity as well as compressibility of firn show a strong dependency on the ice volume fraction. We present the following model:

- Solution of full compressible Stokes-system with thermo-mechanical coupling (material properties dependent on temperature)
- Limited solution of temperature field with respect to negative values of the homologous temperature
- Introduction of firm-properties [1] into dynamics

The model is applied to the Gorshkov crater glacier at Ushkovsky volcano, Kamchatka [2]. Simulation results assuming a steady state of the glacier’s dynamic/thermodynamic properties are presented.

### Governing equations

**Volume balance:**

$$\varepsilon_{i i} + \kappa_{\alpha \beta} = 0$$

**Stokes equation:**

$$-p_{ij} + s_{ij} = -\rho c_{\alpha} D g_i$$

**Heat Transfer:**

$$\rho c_{\alpha} D c \left( \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} \right) = (\kappa T)_{,i}$$

**Dating (age of ice):**

### Constitutive relations

$$\dot{\varepsilon}_{i j} = 2 \eta \cdot (\dot{\varepsilon}_{i j} - \delta_{i j} \dot{\varepsilon}_{k k}/3)$$

$$\eta = (2E(A(T)))^{-1/n} \left( \frac{\dot{e}_{D} \dot{e}_{D}}{\dot{e}_{D}} \right)^{(1-n)/n}$$

$$A(T) = A_0 e^{-Q/R T}$$

$$\kappa = \frac{b}{(1+c_1 D + c_2 D^2) \cdot \dot{c}_1 e^{-\gamma T[K]}}$$

Nomenclature:

- $$\varepsilon_{i j}$$ = i-th velocity component, $$T$$ = temperature, $$p$$ = dynamic pressure, $$D = \rho/\rho_{w} = [0, 1]$$ = relative density, $$A_i =$$ age of ice, $$s_{ij}$$ = deviatoric stress tensor component, $$\dot{\varepsilon}_{i j}$$ = strain-rate tensor component, $$\eta$$ = viscosity, $$A(T)$$ = Arrhenius factor, $$E$$ = enhancement factor, $$\kappa$$ = heat capacity, $$g_i$$ = i-th component of acceleration due to gravity, $$T$$ = temperature, $$\dot{Q} = T_0 - \beta p$$ = pressure melting point, $$Q$$ = activation energy, $$R$$ = universal gas constant, $$\beta$$ = Clausius-Clapeyron coefficient, $$L$$ = latent heat, $$\dot{c}_1 = 570 \text{kg m}^{-2} \text{a}^{-1}$$.

- Averaged accumulation mass flux, $$J_{\text{w}}$$ = reference basal melting rate (obtained from initial simulation run).

### Operating conditions

Imprinted relative density distribution as a function of the flow depth, $$d$$ (from bore-hole K2):

$$D = 1 - 0.55 \cdot e^{-0.038 \cdot d [m]}$$

The bedrock topography of the Gorshkov crater. The white isolines show levels of constant vertical coordinate, $$z$$ in meters a.s.l., with an offset of $$\Delta z = 25 \text{ m}$$. The color texture applied on the bedrock surface shows the local values of the imprinted relative density, $$D$$. The white dot indicates the location of the bore-hole K2.

### Boundary conditions

**free surface:**

$$T = 256.56 \text{K} (-16.6^\circ \text{C}), s_{ij} = 0, A = 0$$

**outflow:**

$$T_{,i} n_i = 0, v_i n_i = \left( \int j_{,i} dA - \int j_{,i} dA \right) / \int \rho dA$$

**bedrock:**

$$\rho_{\text{ice}} D v_i |_{T > 0} = \frac{q_{\text{w}} - \kappa T_{,i} n_i}{L}$$

### Numerical methods

Implementation using open source FE package Elmer [6]:

- Stabilized Method [3] for advective-diffusive systems (flow, heat transfer)
- “Contact” problem [4] for heat transfer with $$T \leq T_{\text{pm}}$$
- Discontinuous Galerkin Method [5] for advection-reaction type systems (dating, evolution of relative density)

### Parameter variation

**Variation of heat flux at bedrock**

$$q_{\text{w}} = q_{\text{min}} + \left( q_{\text{max}} - q_{\text{min}} \right) \cdot \frac{2 - 2 \sin\left( \frac{\pi}{2} \right)}{2 m} \text{ W m}^{-2}$$

(m3): $$m = 3$$, (m4): $$m = 4$$ (reference), (m5): $$m = 5$$

**Variation of firn/ice rheology**

(m4) reference, compressible firn: $$a(D), b(D), E = 1/3$$

(a) incompressible firn: $$b \rightarrow b = 0$$

(b) porous ice ($$a \rightarrow a = 1, b \rightarrow b = 0, E \rightarrow E/D$$)

(c) pure ice ($$a \rightarrow a = 1, b \rightarrow b = 0, E \rightarrow E = 1$$)

### Results

- Temperature, $$T$$, age, $$A$$, and velocity, $$v_i$$, $$v_z$$, profiles over depth, $$d$$, at the K2 bore-hole position.
- Basal melting velocity ($$m_4$$) and levels for $$T \geq T_{\text{pm}} (m4,m5)$$. cases (m3), (a), (b), (c) all show completely temperate base.

### Discussion and outlook

Firn compressibility has to be taken into account. Else, there is too less downward convection of cold/young firn from the free surface, leading to temperate/too old ice in the lower parts of the glacier. Suggested future work:

- Introduction of a mass balance for $$D$$
- Extension of computational domain beyond outflow

### References


[4] Elmer module, contact authors for detailed information.
